Perspectives: Quantum Mechanics on Phase Space[1]

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The basic ideas in the theory of quantum mechanics on phase space are illustrated through an introduction of generalities, which seem to underlie most if not all such formulations and follow with examples taken primarily from kinematical particle model descriptions exhibiting either Galileian or Lorentzian symmetry. The structures of fundamental importance are the relevant (Lie) groups of symmetries and their homogeneous (and associated) spaces that, in the situations of interest, also possess Hamiltonian structures. Comments are made on the relation between the theory outlined and a recent paper by Carmeli, Cassinelli, Toigo, and Vacchini.

1. INTRODUCTION

The formulation of quantum mechanics on phase space, having origins as early as the 1930s (Weyl, 1928; Wigner, 1932), underwent something of a resurgence in the late 1970s and early 1980s. A number of concepts, tools, and elements introduced in the 1950s and 1960s in the theory of quantum measurement (operator-valued measures with non-projector values being perhaps the most significant), which today play an indispensible role in the context of quantum computation and quantum information, have played an equally critical role in theories of quantum mechanics on phase space. The concepts of *positive operator-valued measure* (*POVM*) and *informational completeness* (of a collection of observables) are especially worth mentioning in the phase-space theories of quantum mechanics.

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The present paper is an amalgam of the talks by the two authors, and is aimed at illustrating the basic ideas in the theory of quantum mechanics on phase space through a "gentle" introduction of generalities, which seem to underlie most, if not all, such formulations.

As might be expected in any treatment of particle models embodying non-relativistic or relativistic kinematics, the structures of fundamental importance are the relevant (Lie) groups of symmetries and their respective homogeneous (and associated) spaces which, in the situations of interest, also possess Hamiltonian structures.

Quantum mechanical systems are *characterizable* in terms of their kinematical symmetries that, in their most basic form, are either Galileian or Lorentzian, depending whether the system is required to incorporate non-relativistic or relativistic principles. Since in quantum theory, the description (and construction) of what may be called *multi-particle Hilbert spaces* derives (by suitable tensor products) from *single-particle* formulations, one concentrates on these elementary systems. In Wigner's (1939) formulation in the relativistic context and Lévy-Leblond's (1963) non-relativistic counterpart, these are associated with irreducible, unitary representations on the Hilbert space of the system of the appropriate kinematic group: the Lorentz group or the Galilei group, or the related inhomogeneous group (actually the 11-dimensional *extended* version in the case of the Galilei group in consequence of the fundamental work of Bargmann (1954)). In both situations, the *physical* elementary systems are determined by two real parameters, namely non-negative mass m and spin j taking values that are non-negative half-integer multiples of Planck's fundamental constant. It is now the *physical interpretation* that provides the guide through consideration of the *classical mechanical* phase spaces associated with the *elementary*, *irreducible particle models*. Since the pioneering works of Souriau (1970) and Kostant (1970), it has been recognized that it is the symplectic homogeneous spaces of the appropriate kinematical group that provide the correct phase-space descriptions, and that a particular *model* of the phase-space picture (that embodies the covariance symmetry of the kinematical group) arises either from the *co-adjoint* representation of the group on the dual of its Lie algebra or suitable extensions of the co-adjoint representation approach to take account of topological aspects such as non-trivial group cohomology as in the case of the inhomogeneous Galilei group.

These assumptions are taken to be basic, and one proceeds to note the consequences.

Note that the phase-space formulation of quantum mechanics has little if anything to do with the theory of *geometric quantization* that seeks, through the use of complex polarizations, to reduce the phase-space description to one involving a locally-Poisson-commuting collection of basic coordinates. "Quantum Mechanics on Phase Space" is, in contrast, prepared to accept the need to "live" with phase

space as a fundamental aspect of the description and not attempt to derive it or do away with it from a purely space-time-based approach.

2. PHASE SPACES AND GROUPS

From *classical* experiments, one learns that classical (Newtonian) equations of motion are invariant under translations, boosts (relative velocity transformations between inertial [Galileian] reference frames), and rotations. Prior to 1887, these were invariably viewed to generate the group of Galileian transformations on space-time. However, since the Michaelson and Morley (1886, 1887) experiment, and the subsequent analysis of Voigt (1887), Lorentz (1886, 1892), and Hertz's (1888) clarification of Maxwell's equations, the analysis of FitzGerald (1891), Poincaré (1905, 1906), Einstein (1905a,b, 1911, 1916) and Minkowski (1908, 1909, 1910, 1915), these space-time translations, boosts, and rotations were henceforth interpreted as the generators of the group of Lorentz transformations on either energy–momentum space or on space-time. These transformations generated the entire group (known either as the inhomogeneous Lorentz group or the Poincaré group) from those transformations acting on an arbitrarily small neighborhood of any point (i.e. those transformations in an arbitrarily small neighborhood of the identity in the group). Transformations *infinitesimally* near the identity transformation form a vector space (the Lie algebra of the group) on which a non-associative operation (the Lie bracket) is defined. This was already well known at the time to mathematicians (Lie, Poincaré, and others). Thus, we have the following discussion on classical mechanics and classical experiments.

- Classical experiments reveal the relevant kinematical groups.
  The lesson learned through the efforts of mathematicians over the last 50 years is stated as follows.
- Classical mechanics is describable mathematically on a space with a Poisson bracket, a *phase space*, or more particularly on a symplectic manifold, which possesses a closed, non-degenerate 2-form on it. Furthermore, the relevant Galilei or Poincaré group acts on this space in such a way as to preserve the Poisson bracket (acts "symplectically"). A necessary consequence of this set-up is that so-called "conjugate variables" arise naturally; these are coordinates on the phase space which realize the canonical skew-symmetric form of the Poisson bracket.

With the experience of the Galilei and Poincaré groups, one may abstract this formulation to the setting of the action of a Lie group on *any* phase space.

The group G, being a Lie group, possesses an associated Lie algebra $\mathfrak{g}$ that may be thought of as the collection of all left-invariant vector fields on G. There is a formal invertible process of *exponentiation* that associates an element of the group (near the identity) to any element of the Lie algebra sufficiently near the

origin (zero). One may thus go from the Lie algebra to the Lie group, and vice versa. In what follows it is essential that $\mathfrak{g}$ is a finite-dimensional vector space. If $\wedge$ designates the anti-symmetric tensor product on $\mathfrak{g}$ then one may form the skew-symmetric tensor algebra $*(\mathfrak{g})$ over $\mathfrak{g}$ consisting of elements of various types, namely: $\mathbb{R}, \mathfrak{g}, \mathfrak{g} \wedge \mathfrak{g}, \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$, etc. Let their duals be denoted by $\mathfrak{g}^*$, etc. and note that $\mathfrak{g}^*$ may be thought of as the collection of all left-invariant 1-forms on $G, \mathfrak{g}^* \wedge \mathfrak{g}^*$ as the left-invariant 2-forms on G, and so on. One defines the coboundary operator δ

$$\mathbb{R} \xrightarrow{\delta_0} \mathfrak{g}^* \xrightarrow{\delta_1} (\mathfrak{g} \wedge \mathfrak{g})^* \longrightarrow \cdots$$

as follows. Let (A_i) be a basis of $\mathfrak{g}$ and let (ω^i) be the associated dual basis of $\mathfrak{g}^*$ so that $\omega^i(A_j) = \delta^i_j$. The structure constants of $\mathfrak{g}$, defined relative to the basis (A_i), are determined by the Lie bracket relations: $[A_i A_j] = \Sigma_k C^k_{ij} A_k$. The $\mathbb{R}$ in the aforementioned sequence can be considered to be the collection of left-invariant functions on the group G, which is assumed to be connected, so that the $\mathbb{R}$ may be thought of as the left-invariant 0-forms f on the group. We define

$$\delta_0 f = 0$$

as an element of $\mathfrak{g}^*$. Now, thinking of the ω^i as left-invariant 1-forms one finds that the Maurer-Cartan equations hold: $d\omega^k = -\frac{1}{2} \Sigma_{i,j} C^k_{ij} \omega^i \wedge \omega^j$. We then define

$$\delta_1 \omega^k = -\frac{1}{2} \sum_{i,j} C^k_{ij} \omega^i \wedge \omega^j$$

recognizing that this 2-form is actually in $(\mathfrak{g} \wedge \mathfrak{g})^*$. One extends this expression for δ_1 linearly and thereby obtains the linear map $\mathfrak{g}^* \xrightarrow{\delta_1} (\mathfrak{g} \wedge \mathfrak{g})^*$. Making use of the skew-derivation property for δ_2

$$\delta_2(\lambda \wedge \mu) \equiv (\delta_1 \lambda) \wedge \mu - \lambda \wedge (\delta_1 \mu),$$

for $\lambda, \mu \in \mathfrak{g}^*$. one defines δ inductively.

Letting

$$Z^2(\mathfrak{g}) \equiv \{\omega \in (\mathfrak{g} \wedge \mathfrak{g})^* \mid \delta_2(\omega) = 0\}$$

denote the space of closed, left-invariant 2-forms on G, for $\omega \in Z^2(\mathfrak{g})$, define

$$h_\omega \equiv \{\xi \in \mathfrak{g} \mid \omega(\xi, \cdot) = 0\}.$$

Then h_ω is a Lie sub-algebra of $\mathfrak{g}$ and h_ω determines, by exponentiation, a subgroup H_ω of G. Supposing that H_ω is a closed subgroup of G,

$$\Gamma \equiv G/H_\omega$$

is a manifold. That it is a symplectic manifold (of even dimension equal to $2m$ for some integer m) follows from the fact that the 2-form ω, when factored by

its kernel, is the pull-back of a non-degenerate closed 2-form on G/H_ω. That it is a symplectic G space follows because G acts on G/H_ω by left multiplication on left cosets: $gx = g(g_1 H_\omega) = (gg_1)H_\omega$, where $x = g_1 H_\omega$ for some g_1 in G. Since $\Gamma \equiv G/H_\omega$ is a symplectic manifold, it naturally possesses a left-invariant Liouville measure μ equal to the mth exterior power of ω.

The following result (Theorem 25.1 of Guillemin and Sternberg, 1991) captures the essence of the need for the construction outlined earlier and is sufficient for our purposes, but only in the context of *single-particle* kinematics.

Theorem 1.   *Any symplectic action of a connected Lie group G on a symplectic manifold M defines a G morphism, $\Psi : M \to Z^2(\mathfrak{g})$. Since the map Ψ is a G morphism, $\Psi(M)$ is a union of G orbits in $Z^2(\mathfrak{g})$. In particular, if the action of G on M is transitive, then the image of Ψ consists of a single G orbit in $Z^2(\mathfrak{g})$.*

In the case of the inhomogeneous Lorentz group the co-adjoint orbit construction is sufficient, whereas in the case of the inhomogeneous Galilei group one must consider the symplectic cohomology groups $H^1(\mathfrak{g})$ and $H^2(\mathfrak{g})$ which are both non-trivial. One may consult Section 25 of Guillemin and Stenberg (1991).

- In this fashion, one obtains ALL the *single-particle* symplectic spaces on which G acts symplectically and transitively. In consequence one has a unified mathematical picture of kinematics in the two fundamental cases (Galileian and Lorentzian) of relevance to one-particle physics. Multi-particle kinematics is then described by a phase space that is a Cartesian product of the single-particle phase spaces with symplectic form equal to the "sum" of the symplectic forms on each of the single-particle factors. This is the "méthode de fusion" (Souriau, 1970). In other words, starting from the symplectic action of a group on classical single-particle phase space, one obtains all the phase spaces (single- or multi-particle) on which G acts symplectically, in a physically meaningful way.
- The coordinates on each of these spaces can be sorted into the momentum, position, and rotation coordinates for massive particles, or the frequency, position, and helicity coordinates in the case of the zero mass particles. In answer to the question: "Where does one get these canonical coordinates?" asked by David Finkelstein (1997) this discussion provides at least a partial answer.
- It is emphasized that the same procedure will work for any (connected) Lie group. Thus, results for the Heisenberg group, the affine group, the de Sitter group, etc. have been obtained.

3. HILBERT SPACE ASSOCIATED TO PHASE SPACE

Having chosen $\omega \in Z^2(\mathfrak{g})$ and obtained $\Gamma = G/H_\omega$ and μ, one may form $L_\mu^2(\Gamma)$, which is a Hilbert space on which one may represent G by unitary operators $V(g)$

$$[V(g)\Psi](x) \equiv \Psi(g^{-1}x)$$

for $\Psi \in L_\mu^2(\Gamma)$. Note that it may be necessary in some situations to extend the representation above by incorporating a phase factor.

One may define an operator $A(f)$, for all μ-measurable f, by

$$[A(f)\Psi(x) \equiv f(x)\Psi(x)] \;.$$

These operators on $L_\mu^2(\Gamma)$ have, in the case where the f are characteristic functions $\chi(\Delta)$, the clear classically-motivated interpretation of localization observables in the phase-space region Δ. The collection of quantum mechanical observables includes non-commuting operators and hence must contain operators other than operators of the form $A(f)$. It will become evident that $L_\mu^2(\Gamma)$ is not a Hilbert space of fundamental importance to the description of quantum mechanical models of elementary (i.e., irreducible), single-particle systems, but, that it is reducible into a direct sum (or integral) of such irreducible spaces.

4. QUANTUM MECHANICAL REPRESENTATION SPACES

In the case of the inhomogeneous Galilei and Lorentz groups, the "Mackey Machine" (Mackey, 1952, 1953) and the earlier Wigner (1939) classification are well known to produce all continuous, irreducible, unitary Hilbert space representations and that these are characterized by the Casimir invariants in the universal enveloping algebra of the Lie algebra. These Casimir elements are identifiable as the physical quantities of rest mass and spin (or helicity in the mass-zero case). For the inhomogeneous Galilei group one had to wait until the analysis of Lévy-Leblond (1963) to achieve a similar picture physically characterized by mass and spin.

It is the case that the well-known irreducible representation spaces for both the Galilei and Lorentz group are "single-particle Hilbert spaces" in the usual language of physics, and are Hilbert spaces of square-integrable functions over single-particle momentum-energy spaces.

What we will see later is that the correspondence between "irreducible" and "single-particle" is best elucidated within the Hilbert space constructed-over-phase-space framework.

In what follows, U will usually denote an irreducible unitary representation of G on an irreducible representation space, usually denoted $\mathcal{H}$.

5. PHASE SPACE AND QUANTUM REPRESENTATIONS

The critical idea is the following.

One wishes to define a linear transformation W^η from $\mathcal{H}$ to $L^2_\mu(\Gamma)$ by

$$[W^\eta(\varphi)](x) \equiv \langle U(\sigma(x))\eta, \varphi \rangle$$

for $x \in \Gamma = G/H_\omega$, for all $g \in G$, and for all $\varphi \in \mathcal{H}$, where η is a vector in $\mathcal{H}$ and where σ is a (Borel measurable) section

$$\sigma : G/H_\omega \longrightarrow G.$$

The reason to define such a map is that one seeks to encode the entire content of the state vector $\varphi \in \mathcal{H}$ into a complex-valued function on the phase space Γ in a manner that is reversible. The goal is to be able to reconstruct the state from the complex numbers $[W^\eta(\varphi)](x)$ which encode it.

To ensure that the image of W^η actually lies in $L^2_\mu(\Gamma)$ one must exercise some care in the choice η. Accordingly,

(a) one selects and fixes, once and for all, a (Borel measurable) section $\sigma :$ $G/H_\omega \longrightarrow G$;

(b) one chooses a "suitable" resolution generator $\eta \in \mathcal{H}$.

The trick here is to decide what "suitable" means. One says that η is *admissible* with respect to the section σ if

$$\int_\Gamma |\langle U(\sigma(x))\eta, \eta \rangle|^2 d\mu(x) < \infty.$$

Assuming that η is admissible with respect to σ, one says that η is α-*admissible with respect to σ* if in addition to admissibility of η one also has

$$U(h)\eta = \alpha(h)\eta$$

for all h in H_ω, where a α is a one-dimensional representation of H_ω.

If η is α-*admissible with respect to σ* then we have what is needed to properly define the mapping W^η from $\mathcal{H}$ to $L^2_\mu(\Gamma)$ and to carry out the analysis needed to describe states $\varphi \in \mathcal{H}$ by their images $W^\eta(\varphi)$ in $L^2_\mu(\Gamma)$ (Schroeck, 1996).

To illustrate these conditions, consider

- the case of a massive, spinless, relativistic particle (G = Poincaré group) in which one finds (Ali *et al.*, 1988) that η must be rotationally-invariant under $H_\omega = SU(2)$, and square-integrable over $\Gamma \equiv G/H_\omega \cong \mathbb{R}^6 \cong \mathbb{R}^3_{position} \times \mathbb{R}^3_{momentum}$ the classical phase space of a massive, relativistic spinless particle.

- the case of a massive, relativistic particle with non-zero spin ($G =$ Poincaré group) in which one finds (Brooke and Schroeck, 1989, and

Brooke and Schroeck, in preparation) that η must be rotationally invariant about the "spin axis" (but not necessarily invariant under all rotations in $SU(2)$), i.e., invariant under H_ω = double covering of $O(2) \cong$ stabilizer in $SU(2)$ of the spin axis, and square-integrable over $\Gamma \equiv G/H_\omega \cong \mathbb{R}^3_{\text{position}} \times \mathbb{R}^3_{\text{momentum}} \times S^2_{\text{spin}}$ the classical phase space of a massive, relativistic, spinning particle.

Orthogonality relations, which play a prominent role in the representation theory of compact groups, also appear in this approach. Assuming that the vectors $\eta_i \in \mathcal{H}, i = 1, 2$ are α-admissible, one may prove the existence of a unique, positive, invertible operator C such that for all $\varphi_i \in \mathcal{H}$, there holds an "orthogonality relation" of the form (Healy and Schroeck, 1995)

$$\int_\Gamma \langle \varphi_1, U(\sigma(x))\eta_1 \rangle \langle U(\sigma(x))\eta_2, \varphi_2 \rangle d\mu(x) = \langle C\eta_2, C\eta_1 \rangle \langle \varphi_1, \varphi_2 \rangle.$$

Note, in the case of a compact group, the positive operator C simplifies to a positive *constant*. In fact, this orthogonality relation holds with C a positive constant on any group in which there is satisfied yet another admissibility condition—the *β-admissibility condition*—for an α-admissible vector $\eta \in \mathcal{H}$. An *α-admissible* vector η is said to be *β-admissible* if, when g is any commutator of group elements $\sigma(x)^{-1}\sigma(y)^{-1}\sigma(x)\sigma(y)$, then $U(g)\eta = \beta(x, y)\eta$ for some scalar function $\beta(x, y)$. The β-admissible condition holds in the case of the inhomogeneous Galilei group, but not for the Poincaré group, suggesting that Poincaré group orthogonality relations are not expressible with the right-hand side of the form $\frac{1}{d}\langle \eta_2, \eta_1 \rangle \langle \varphi_1, \varphi_2 \rangle$ for d a positive constant independent of η_1, η_2, and φ_1, φ_2; if d exists, then

$$1/d = \|\eta\|^{-4} \int_\Gamma |\langle U(\sigma(x))\eta, \eta \rangle|^2 d\mu(x).$$

For the sake of simplicity, we denote the closure of the image of W^η by $W^\eta(\mathcal{H}) \subset L^2_\mu(\Gamma)$. Let P^η denote the canonical projection

$$P^\eta : L^2_\mu(\Gamma) \longrightarrow W^\eta(\mathcal{H})$$

and denote by $A^\eta(f)$ the mapping (Schroeck, 1996)

$$A^\eta(f) \equiv [W^\eta]^{-1} P^\eta A(f) W^\eta : \mathcal{H} \longrightarrow \mathcal{H}.$$

This is a plausible candidate for the quantum mechanical operator that corresponds to the classical observable f. For example, for the Heisenberg group and for $\eta =$ the ground state wave function of the harmonic oscillator, then $A^\eta(q) = Q =$ the position operator, and $A^\eta(p) = P =$ the momentum operator.

One can prove (Schroeck, 1996) that $A^\eta(f)$ has an operator density $T^\eta(\cdot)$:

$$A^\eta(f) = \int_\Gamma f(x)T^\eta(x) \, d\mu(x),$$

$$T^\eta(x) \equiv |U(\sigma(x))\eta\rangle\langle U(\sigma(x))\eta|,$$

and that

$$A^\eta(1) = 1.$$

With this set-up one can make a number of remarks:

1) Let ρ denote any quantum density operator; i.e., ρ is non-negative and has trace one. Then one may write $\rho = \Sigma \rho_i P_{\Psi_i}$, the ψ_i forming an orthonormal set and P_{ψ_i} denoting the corresponding projection. Now, using the interpretation of $|\langle U(\sigma))\eta, \psi_i\rangle|^2$ as the transition probability from ψ_i, to $U(\sigma(x))\eta$ one has the quantum expectation value given by

$$\mathrm{T_r}(\rho A^\eta(f)) = \sum_i \rho_i \int_\Gamma f(x)|\langle U(\sigma(x))\eta, \psi_i\rangle|^2 d\mu(x);$$

i.e., the sum over the transition probabilities (Schroeck, 1996). For example, when using a "screen" to detect a particle in a vector state given by ψ, one idealizes the detector (the screen) as a multi-particle quantum system consisting of identical sub-detectors. In a fixed laboratory frame of reference a sub-detector is represented by a state vector η whose phase-space counterpart $W^\eta \eta$ is peaked about a reference phase-space point, which may be referred to as "the origin." For a fixed space-time reference frame, one may "position" a detector at all "points" of space-time (space-time events) exactly as Einstein located rods and clocks. Of course, one must now position mass spectrometers (devices that measure rest-mass in their own rest frames) and Stern–Gerlach devices at all space-time events in addition to rods and clocks. As Einstein imagined that the rods and clocks were also equipped (at all space-time coordinate events) in all inertially-related space-time reference frames, so must we imagine that our inertially-related space-time reference frames carry identical mass spectrometers and Stern–Gerlach devices in addition to rods and clocks (boosted relative to the rest "laboratory" frame). So, instead of rods and clocks situated at each space-time event and at rest in inertially-related (uniformly moving) rest frames, we must add to that *imagery* a more elaborate set of apparati. For a fixed value of momentum p there are infinitely many pairs (m, u) such that $p = mu$; of course the momentum does not alone characterize the uniform relative velocity (boost) represented by p - one requires also the rest-mass m. The totality of all such "placements" of detectors constitutes the phase-space distribution of detectors - the classical phase-space frame analogous to the classical space-time (Lorenz) frame (of rods and clocks). Thus, the complete detector is composed of sub-detectors each located at different "positions" (points of Γ). The sub-detector located at "position" $x \in \Gamma$, obtained from η by a kinematical placement procedure (with the same intent as Einstein's placement

of identical rods and clocks at all points of space-time), is $U(\sigma(x))\eta$. Since the probability that ψ is captured in the state given by $U(\sigma(x))\eta$ is $|\langle U(\sigma(x))\eta, \psi\rangle|^2$, the formula for the expectation is justified. *One cannot improve on this procedure when measuring, by quantum mechanical means, the distribution of the particle.*

2) Since $T^\eta(x) \geq 0$ and $A^\eta(1) = 1$, then $\rho_{\text{class}}(x) \equiv \text{Tr}(\rho T(x))$ is a classical (Kolmogorov) probability function (Schroeck, 1996). Consequently,

$$\text{quantum expectation} = \text{Tr}(\rho A^\eta(f))$$

$$= \int_\Gamma f(x)\text{Tr}(\rho T^\eta(x))d\mu(x)$$

$$= \int_\Gamma f(x)\rho_{\text{class}}(x)d\mu(x)$$

$$= \text{classical expectation.}$$

3) Since the operators $A^\eta(f)$ enjoy the feature of the same expectation as the "classical" observables f, one might ask whether these operators are sufficient to distinguish states of the quantum system.

Definition 1. Prugovečki, 1977: A set of bounded self-adjoint operators $\{A_\beta | \beta \in I, I$ some index set$\}$ is informationally complete iff for all states ρ, ρ' such that $Tr(\rho A_\beta) = Tr(\rho' A_\beta)$ for all $\beta \in I$ then $\rho = \rho'$.

Example 1. Prugovečki, 1977: In spinless quantum mechanics, the set of all spectral projections for position is not informationally complete. Neither is the set of all spectral projections for momentum, nor even the union of them.

The $\{A^\eta(f) | f$ is measurable$\}$ (or, equivalently $\{T^\eta(x) | x \in \Gamma\}$ is known to be informationally complete in a number of cases:

a) spin-zero massive representations of the Poincaré group (Ali *et al.*, 1988);

b) mass-zero, arbitrary helicity representations (Brooke and Schroeck, 1996) of the Poincaré group;

c) the affine group (Healy and Schroeck, 1995);

d) the Heisenberg group (Schroeck, 1996);

e) massive representations (Ali and Prugovečki, 1986; Schroeck, 1996) of the inhomogeneous Galilei group;

f) massive, non-zero spin representations (Brooke and Schroeck, in preparation) of the Poincaré group are being investigated.

4) If $\{A_\beta | \beta \in I\}$ is informationally complete then any bounded operator on $\mathcal{H}$ may be written as (a closure of) integrals over the set I (Busch, 1991).

5) When we specialize $A^\eta(f)$ to $f = \chi(\Delta)$, $\chi(\Delta)$ the characteristic function for the Borel set Δ, then

$$\chi(\Delta) = \text{classical localization in } \Delta \subset \Gamma,$$

$$A(\chi(\Delta)) = \text{operator on } L^2(\Gamma) \text{ localizing } \Delta \subset \Gamma,$$

$$A^\eta(\chi(\Delta)) = \text{operator on } \mathcal{H} \text{ localizing in } \Delta \subset \Gamma.$$

These $A^\eta(\chi(\Delta))$ have several properties (Schroeck, 1996), among them:

a) If Δ is a compact subset of Γ then $A^\eta(\chi(\Delta))$ is a compact operator with spectrum in [0,1],

b) For all $\Delta \subset \Gamma$, $\|A^\eta(\chi(\Delta))\| \leq \mu(\Delta)$.

Some consequences of this set-up appear in subsequent sections.

6. COMMENT ON A PAPER BY CARMELI, CASSINELLI, TOIGO, AND VACCHINI

The paper in question is: *A complete characterization of phase-space measurements* (Carmeli *et al.*, 2004).

The following remark holds also for the case of non-zero spin for either Galileian or Lorentzian massive one-particle situations, but is simplified to the case treated in the paper by Carmeli *et al.* (2004) namely, spin-zero.

In the present formalism, by making use of an isometry (the Wigner transform) from the Hilbert space of square-integrable functions on three-dimensional space (the irreducible representation space of the canonical commutation relations [of the Weyl–Heisenberg group]) to a subspace of the Hilbert space of square-integrable functions on physical six-dimensional phase space, one may determine the irreducible unitary subrepresentations of the inhomogeneous Galilei and Lorentz groups arising from a *resolution generator* via the Wigner transform (Prugovečki, 1978a,b). See also Ali and Prugovečki (1986a), and Brooke (1987). In this way, the invariance of the density matrix under rotations (as posited by Carmeli *et al.*, 2004) results from the invariance of the resolution generator under rotations. The advantage of the resolution generator view is that the localization operators on phase space arise naturally within the unified theory in which the quantum mechanical Hilbert space is constructed directly from the classical phase space. Moreover, the same construction applies in the relativistic setting, and furthermore, in the non-zero-spin situations (Ali *et al.*, 1988).

Carmeli *et al.* (2004) treat spin-zero, massive, one-particle, non-relativistic quantum mechanics and obtain a characterization of phase-space measurements. The authors state in Section 3: "In the present, section, we characterize all the

phase-space measurements of a non-relativistic particle of mass m. For the sake of simplicity we restrict to the spinless case, the extension to the general case being straightforward." To the contrary, it is our experience that the non-zero spin case is not as straightforward as is often claimed *if*, as outlined earlier, one is expected to introduce the spin through phase-space and group theoretic considerations rather than by *ad hoc* constructions. They have it backwards from the present point of view in the sense that the phase space is determined by the kinematic group. They do not take into account the fact that the spin and the angular momentum are intertwined. Their proof of their principal result may not be valid in the relativistic situation where, as was mentioned in Section 5, in the Poincaré case one should not expect an orthogonality relation with the operator C a constant equal to $1/d$.

7. UNCERTAINTY RELATIONS AND CHANNEL CAPACITY THEOREM

Consider the measuring instrument (represented by η) to be fixed. Since $A^\eta(\chi(\Delta))$ is a compact operator for compact Δ (Schroeck, 1996) let it have eigenvalues λ_i with corresponding eigenvectors ψ_i:

$$A^\eta(\chi(\Delta))\psi_i = \lambda_i \psi_i, \quad 0 \le \lambda_i \le 1.$$

One says that ψ_i is *localized in* Δ if λ_i is *close* to 1 (say $\lambda_i > 1 - \epsilon$). When localized (in Δ) by $A^\eta(\chi(\Delta))$, ψ_i will be attenuated by the factor λ_i. One has from the earlier equation that

$$\lambda_i = \mathrm{Tr}[A^\eta(\chi(\Delta))|\psi_i\rangle\langle\psi_i|]$$
$$= \int_\Delta \mathrm{Tr}[T^\eta(x)|\psi_i\rangle\langle\psi_i|]d\mu(x) \le \mu(\Delta)$$

and, in fact, more strongly that

$$\sum_i \lambda_i = \mathrm{Tr}[A^\eta(\chi(\Delta))] \le \mu(\Delta)$$

which is a sharper upper bound on the λ_i when $\mu(\Delta)$ is small, in particular if $\mu(\Delta) \le 1$, in units of $\hbar$, when the phase space is two-dimensional.

This exemplifies the following version of "the uncertainty relation": it is impossible to localize a physical quantum system in an arbitrarily small volume in phase space (Feynman *et al.*, 1963). The following result is useful to establish this uncertainty relation. If Δ has a smooth boundary, one can prove (Schroeck, 1996) that the λ_i, are clustered near 1 and 0 with almost no λ_i between ϵ and $1 - \epsilon$, for some $\epsilon > 0$. (For example, take $\epsilon = \frac{1}{n}$, where n is an integer bigger than 3.) If there were N of the λ_i clustered near 1 then $N \approx \Sigma \lambda_i$ near 1 $\le \mu(\Delta)$ which, when $\mu(\Delta)$ is small, requires $N \le 1$.

There are a number of examples from the world of classical mechanics whose analysis is improved by treating it as a quantum mechanical system. Our first example is the channel capacity theorem: "In a time interval from $-T$ to T and in a band width of size Ω, the total number of channels that can pass through the device is $2\Omega T$." This theorem was originally argued to be true by "modifying" the signal and its Fourier transform. But we know mathematically, that a non-zero signal and its Fourier transform may not both have compact supports. Now, if we take the time-frequency space as a phase space and treat the number of channels as the number of orthogonal wave functions ψ_i that can pass though (read as "when localized in") the device without severe attenuation, we can obtain the "$2\Omega T$" result from the earlier analysis.

There are many other subjects that can be profitably analyzed with this phase-space formalism. It may seem strange to consider some of them as quantum mechanical systems, but that has been done. To list a few: (1) neutron interferometry, (2) single slit experiments, (3) Stern–Gerlach devices, (4) CT scans, (5) NMR, (6) MRI, (7) holography, (8) bat echo-location, (9) the olfactory system of dogs, (10) neural networks in the brain, (11) geologic exploration, (12) clearing mine fields (which is being investigated by someone at this conference), (13) radar, etc. Many of these and others have been investigated and results may be found in Busch *et al.* (1995) and Schroeck (1996).

To make *obvious* the point that the phase-space perspective is necessary, take the system by which one *sees*. One's brain creates a display of both the image in 3-space and in color. The phase space of the photon of either positive or negative helicity is topologically homeomorphic to $\mathbb{R}^3 \times \mathbb{R}^+ \times S^2$ where the $\mathbb{R}^3$-factor is the position space, the $\mathbb{R}^+$-factor is the frequency space, and the S^2-factor is the space of rotations in the momentum space. See Brooke and Schroeck (1996). Thus, you may place an instrument at any point in configuration space, turn it so it points in any direction, and then measure the frequency (or wavelength) and helicity. When one looks in one direction (possibly aided by polarized glasses), the brain makes measurements in phase space!

8. EFFECT ALGEBRAS

To bring the discussion closer to other of the topics of this conference, we begin with three definitions.

Definition 2. An operator A in any Hilbert space is an ***effect*** if A is self-adjoint, non-negative ("positive"), and bounded above by 1.

Definition 3. An ***effect algebra*** E is a set containing 0 and 1, with a partial binary operation $\oplus$ on E satisfying (i) if a, b, and $a \oplus b \in E$, then $b \oplus a \in E$ and $a \oplus b = b \oplus a$; (ii) if a, b, c, $b \oplus c$, $a \oplus E$, then $a \oplus b$, $(a \oplus b) \oplus c \in E$ and

a ⊕ (b ⊕ c) = (a ⊕ b) ⊕ c; (iii) $\forall a \in E, \exists! a' \in E$ such that $a \oplus a' = 1$; (iv) if $a \oplus 1 \in E$, then $a = 0$.

The set of all effects in a Hilbert space is *an* effect algebra. As one will see, in a Hilbert space it is not the only one.

If A is an effect on $\mathcal{H}$ and ρ is any density operator on $\mathcal{H}$, then $0 \leq \mathrm{Tr}(A\rho) \leq 1$. Thus, one may view $\mathrm{Tr}(A\rho)$ as the expected value of A in state ρ.

Definition 4.   A positive operator-valued measure (POVM) is a mapping A from any σ-algebra Σ on any set Γ to the non-negative ("positive") self-adjoint operators on $\mathcal{H}$ such that: (i) $A(\Gamma) = 1$, (and $A(\phi) = 0$), (ii) for every countable collection $\{\Delta_i\}$ of disjoint measurable sets ($\Delta_i \in \Sigma$), $A(\cup_i \Delta_i) = \Sigma_i A(\Delta_i)$ in the topology of weak operator convergence). A projection-valued measure (PVM) is a POVM in which all $A(\Delta_i)$ are projections.

If, in the formalism of quantum mechanics on phase space, one defines

$$E^\eta \equiv \{A^\eta(f) | 0 \leq f \leq 1, \ f \text{ Borel measurable}\},$$

$$F^\eta \equiv \{A^\eta(\chi_\Delta) | \Delta \text{ Borel measurable on } \Gamma\},$$

with $\oplus$ defined by $A^\eta(f_1) \oplus A^\eta(f_2) \equiv A^\eta(f_1 + f_2)$ when $f_1 + f_2 \leq 1$, $[A^\eta(f)]' \equiv 1 - A^\eta(f)$, etc. (Schroeck, 2001), then one obtains the following theorem.

Theorem 2. $F^\eta \subset E^\eta$; E^η, F^η *are effect algebras;* F^η *generates a POVM which is informationally complete on* $\mathcal{H}$ *for suitable* η.

Remarkably, we have the following:

1) There is no projection in E^η other than 0 and 1 (Schroeck, 2001). Thus, one does not obtain a PVM from either E^η or F^η.
2) E^η is not only an effect algebra, it is also an interpolation algebra, a Riesz decomposition algebra, a lattice ordered effect algebra, a distributive algebra, an MV algebra, and a Heyting algebra. It is not a Boolean algebra (Schroeck, 2004).
3) E^η is not an orthoalgebra. The property $a \wedge a' = 0$ is equivalent to all the $A^\eta(f)$'s being projections, which is ruled out by 1). Included in this are all "finite quantum logics." To approximate these projections, one would need to look at an informationally complete set of the $A^\eta(f)$'s for the f's being just measurable real-valued functions.
4) The η involved is a wave function for the measuring instrument. It is an essential ingredient to achieve a true quantum measurement.

5) According to philosophers and logicians, the logic by which quantum computers should be designed is an MV algebra that is also a Heyting algebra. The remarks above justify this assertion.

6) Taking the $A^\eta(f)$s as the only realistically allowed operators in the theory of quantum computers, then *the theory based on projections* is only an approximation to reality. (Here "approximation" is based on the fact that the $A^\eta(f)$'s are informationally complete.) One must have, at a minimum, a POVM that is not a PVM. Furthermore, that POVM must reflect phase-space variables in some sense. When one makes a finite-dimensional approximation of the Hilbert space to carry out numerical computations, one must make an appropriate choice of a POVM. The same can be said of any numerical computations in quantum theory.

7) Given that the $A^\eta(f)$s are the only realistically allowed operators, one may re-analyze the axioms of "quantum computation" and their consequences under which the "results" of quantum computation are derived. For example, whether Shor's algorithm for factoring large numbers is indeed implementable in any approximate sense should be investigated.

9. CONCLUSION

Any kinematical group may be analyzed in the fashion of quantum mechanics on phase space. Quantum mechanical measurement usually leads to a POVM that is not a PVM, a direct consequence of an inherent spread of the wave function of the particles being measured. The effects of decoherence and measurement inaccuracy are in addition to this inherent imprecision. It is our view that the methods of *quantum mechanics on phase space* must be taken into account in order to express predictions and to analyze experiments in quantum theory; in particular, in order to decide whether or not quantum computers are physically realizable within this framework.

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